Crash Course on Advance Linear Algebra I Session - 1

Special class

Revti Raman Sharma • July 19, 2021

MATHEMATICAL SCIENCES

SPECIAL SESSION

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Description:

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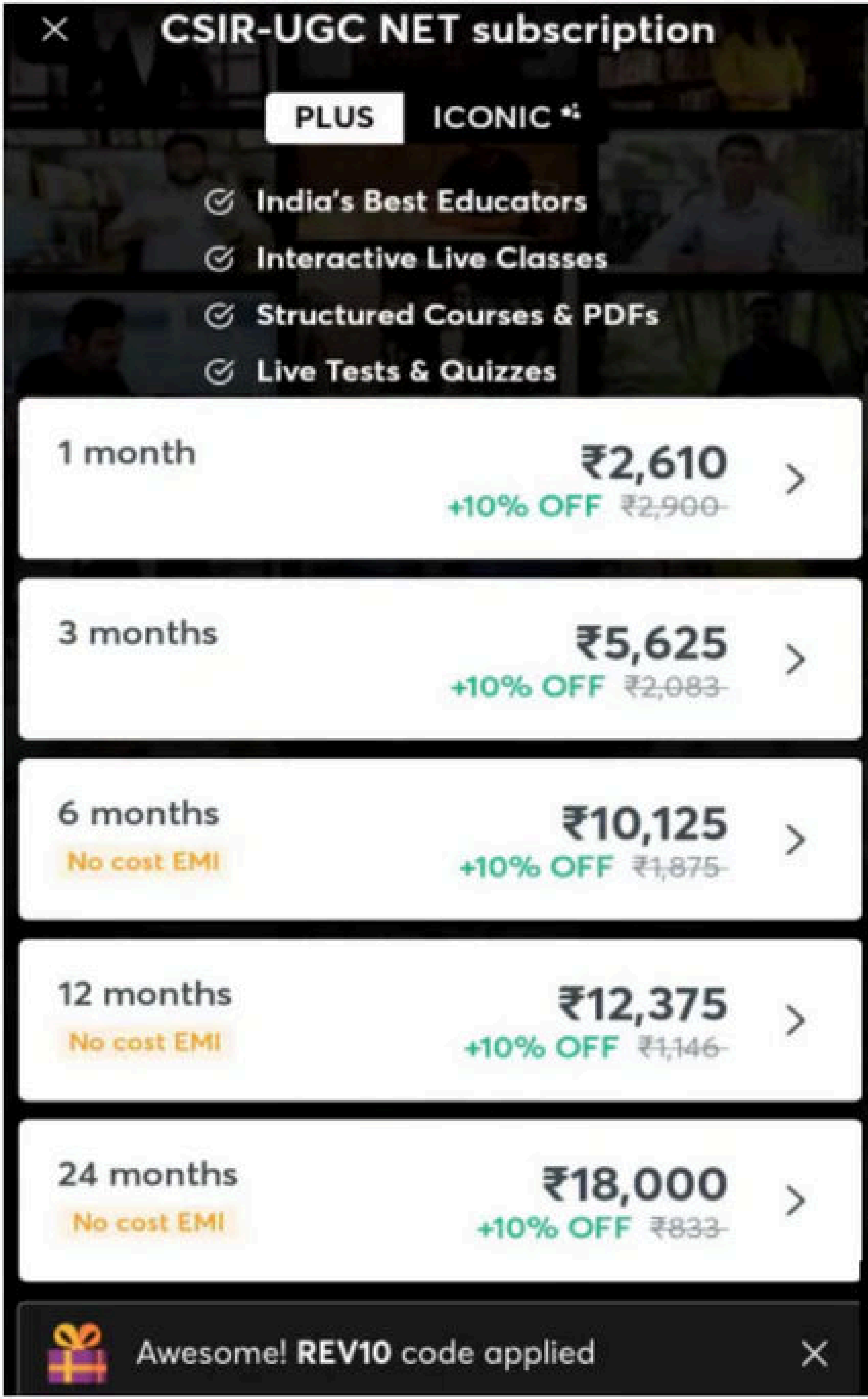

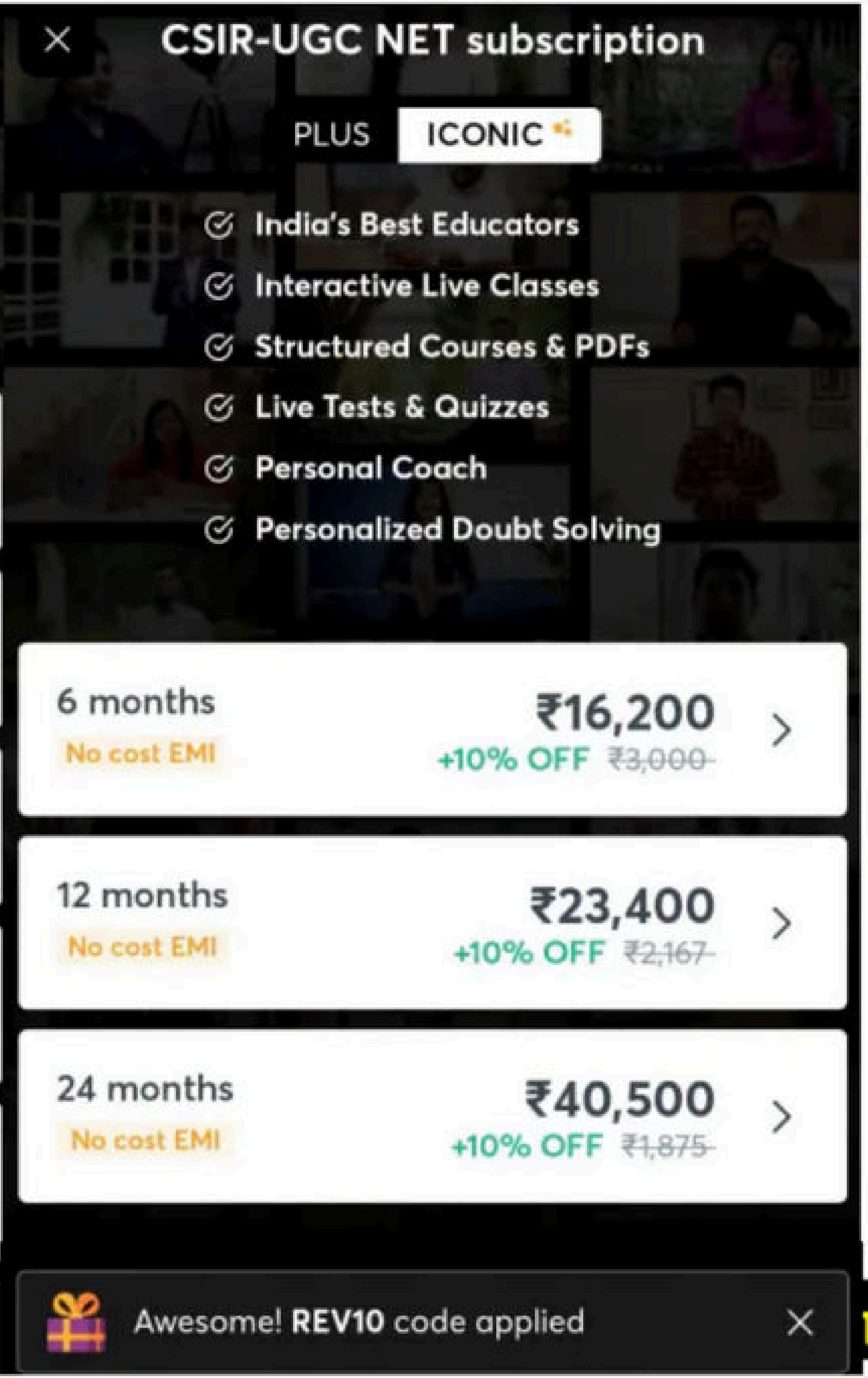

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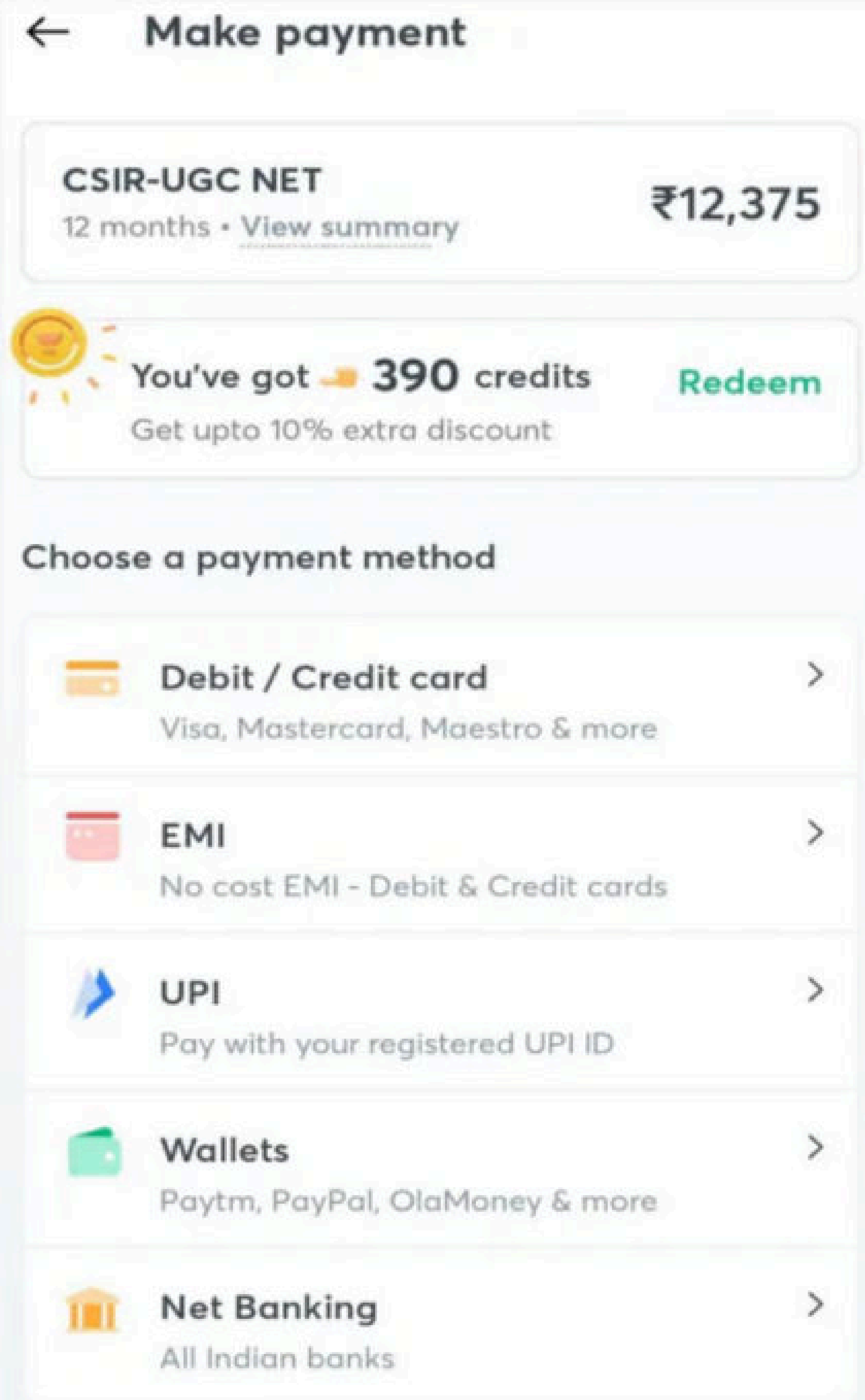

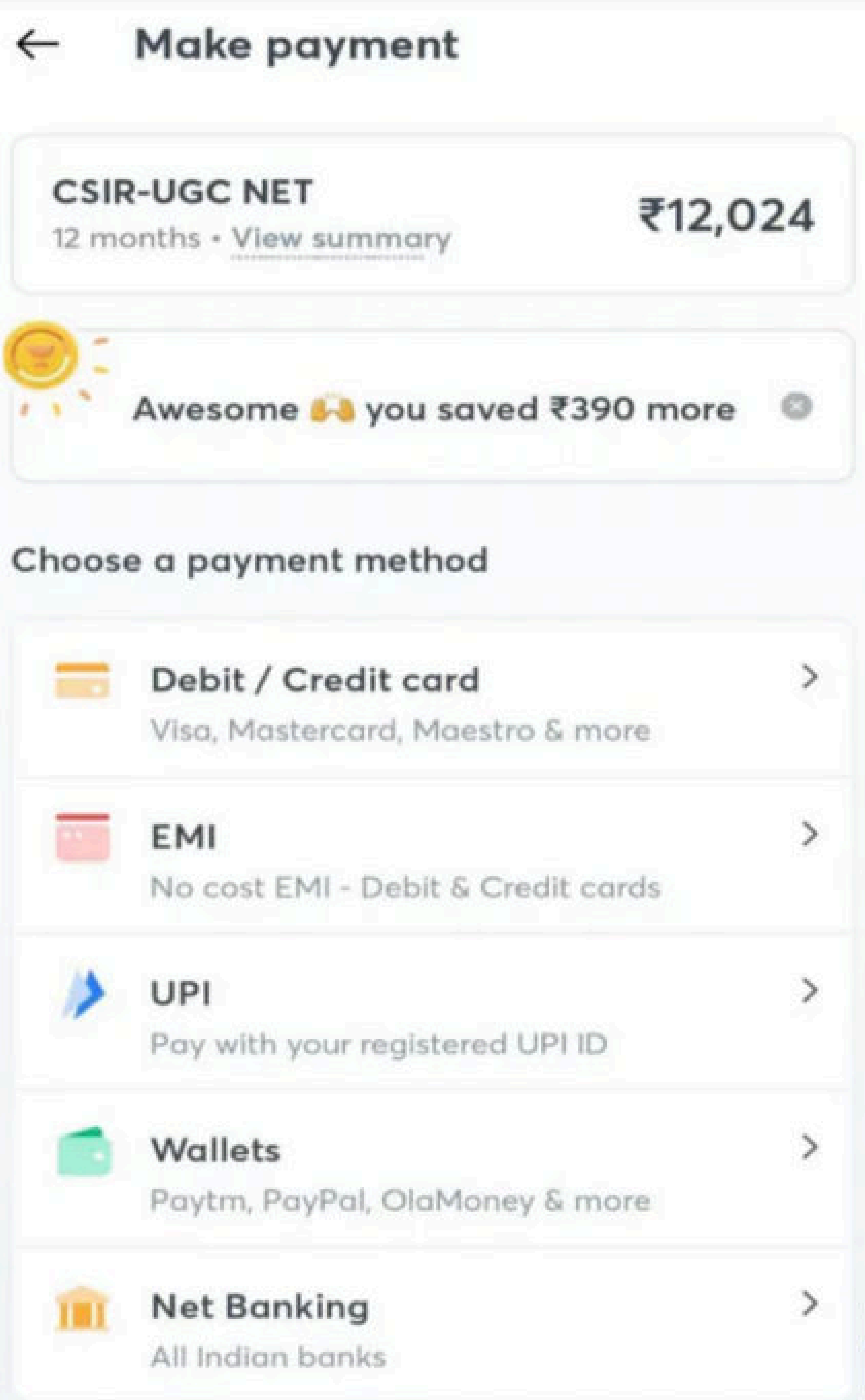

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Contents:

Quadratic Forms

Quadratic Form: An expression q in n variables is called a quadratic form if it is an homogeneous expression of degree two.

Matrix of a Quadratic Form: Let q be a quadratic form in variables $x_1, x_2, \ldots x_n$ over a field F. A $n\times n$ matrix A is said to be matrix of q if

(i) A is symmetric. (ii) $q(X) = X'AX$, where $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Real Quadratic Form: A quadratic form is said to be real if its matrix A is over the real field $\mathbb{R}$ and the column vector X is in the vector space $\mathbb{R}^n$.

Note: We shall confine our study to real quadratic forms only. Therefore, unless otherwise stated, a quadratic form will be understood to be a real quadratic form.

Method: To write the matrix of a quadratic form.

Input: A quadratic form $q(X)$ is given.

Output: Symmetric matrix of $q(X)$.

Working Steps: The matrix of the quadratic form is given by

$$A = \begin{bmatrix} \text{coeff of } x_1^2 & \frac{1}{2}\text{coeff of } x_1x_2 & \cdots & \frac{1}{2}\text{coeff of } x_1x_n \\ \frac{1}{2}\text{coeff of } x_2x_1 & \text{coeff of } x_2^2 & \cdots & \frac{1}{2}\text{coeff of } x_2x_n \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{2}\text{coeff of } x_nx_1 & \frac{1}{2}\text{coeff of } x_nx_2 & \cdots & \text{coeff of } x_n^2 \end{bmatrix}$$

Method: To Find a Quadratic Form When its Matrix is given.

Input: Matrix of a quadratic form $A=\left[a_{ij}\right]_{n\times n}$ is given.

Working Steps: (i) Take $X=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

(ii) Compute X'AX and we get the required quadratic form.

Assignment - 60

1). Which of the following are quadratic forms:

(i) $x^2 + y^2 - xy + yz + 4y$

(ii) $x_1^2 - 2x_1x_2 + x_3^2 - 4x_1x_2x_3$

(iii) $x_1^2 + x_2^2 + 3x_3^2 - 2x_1x_2$

(iv) $x^2 + y^2 + z^2 - 4xy + 2xz + 3xyz + 5$

(v) $x^2 + y^2 + z^2 - 2xy$

(vi) $x_1^2 + x_1x_2 - x_1x_3$

2). Write the symmetric matrices of the following quadratic forms:

(i) $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + x_2x_3 + 4x_1x_3$

(ii) $(x+y)^2 + (z+t)^2$

(iii) $x_1x_2 + 3x_2x_3 + 2x_1x_3$

(iv) $ax^2 + 2hxy + by^2$

(v) $ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx$

(vi) $x_1^2 - 2x_2^2 + 3x_3^2 - 4x_2x_3 + 6x_1x_3$

(vii) $2x^2 + 4y^2 + 3z^2 - 2xy$

(viii) $x_1x_2 + x_2x_3 + x_3x_1$

3). Write the quadratic forms corresponding to following symmetric matrices:

(i) $\begin{bmatrix} 1 & 4 \\ 4 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 4 & 5 & 7 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 5 & 3 \\ 5 & 0 & 4 \\ 3 & 4 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 3 & 4 \\ 3 & 1 & \frac{7}{2} \\ 4 & \frac{7}{2} & 1 \end{bmatrix}$

(v) $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

(vi) $\begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 2 & 0 & 1 \\ 5 & 0 & 3 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix}$

Answers – Assignment - 60

1). (i) No (ii) No (iii) Yes

(iv) No (v) Yes (vi) Yes

2). (i) $\begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & \frac{1}{2} \\ 2 & \frac{1}{2} & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 0 & \frac{1}{2} & 1 \\ \frac{1}{2} & 0 & \frac{3}{2} \\ 1 & \frac{3}{2} & 0 \end{bmatrix}$

(iv) $\begin{bmatrix} a & h \\ h & b \end{bmatrix}$

(v) $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

(vi) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 3 & -2 & 3 \end{bmatrix}$

(vii) $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(viii) $\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$

3). (i) $x_1^2 + 8x_1x_2 + 2x_2^2$

(ii) $x_1^2 + 3x_2^2 + 7x_3^2 + 4x_1x_2 + 8x_1x_2 + 10x_2x_3$

(iii) $2x_1^2 + x_3^2 + 10x_1x_2 + 6x_1x_3 + 8x_2x_3$

(iv) $x_1^2 + x_2^2 + x_3^2 + 6x_1x_2 + 8x_1x_3 + 7x_2x_3$

(v) $ax^2 + by^2 + cz^2$

(vi) $x_1^2 + 2x_2^2 + 3x_3^2 + 4x_4^2 + 10x_1x_3 + 4x_1x_4 + 2x_2x_4 + 4x_3x_4$

Diagonalization

Discriminant of a Quadratic Form: The discriminant of a quadratic form is defined to be the determinant of its matrix i.e. for $q(X) = X'AX$, the discriminant of $q(X)$ is det (A).

Rank of a Quadratic Form: Rank of a quadratic form is defined to be the rank of its matrix.

Singular and Non-singular Quadratic Forms: A quadratic form is said to be singular if its discriminant is zero; otherwise it is called non-singualar.

Def: Let $q(X) = X'AX$ be any quadratic form. A substitution $X = BY$ in it is called a linear transformation of the quadratic form. Further, if B is singular then linear transformation $X = BY$ is called singular and if B is non – singular then it is called non – singular.

Result 1: A quadratic form remains a quadratic form when subjected to a linear transformation.

Result 2: The rank of a quadratic form is invariant under a non-singular linear transformation.

Diagonal Quadratic Form: A quadratic form $q(X) = X'AX$ is called diagonal if matrix A is of the type $\begin{bmatrix} D_r & O \\ O & O \end{bmatrix}$ where D_r is a diagonal matrix of order r.

OR

A quadratic form q in the variables $x_1, x_2, ..., x_n$ is called canonical or diagonal if it is represented as $q = a_1x_1^2 + a_2x_2^2 + ... + a_rx_r^2$ where $r \leq n$.

Diagonalization: The process of transforming a quadratic form into a diagonal form is called diagonalization.

There are many methods to diagonalize a quadratic form. Among them we shall discuss two methods.

Result 3: Every real quadratic form of rank *r* can be reduced to the form $d_1y_1^2 + d_2y_2^2 + ... + d_ry_r^2,\ d_i \neq 0$ by the application of a linear transformation.

OR

Every real quadratic form on a field can be reduced to a diagonal form.

Method: To Diagonalize a Quadratic Form.

Input: A quadratic form.

Output: Equivalent diagonal quadratic form and equations of transformation.

Working Steps:

(i) Find the symmetric matrix A of the given quadratic form.

(ii) Form the augmented matrix $[A \; : \; I]$.

(iii) Check the entry a_{11}. If $a_{11} \neq 0$ its okay but if $a_{11} = 0$ then make is non – zero by suitable row operations and corresponding column operations.

(iv) Make 0's below a_{11} by applying row operations.

(v) Apply the corresponding column operations of step (iv) to make 0's on the right of a_{11}.

(vi) Repeat the above process on $a_{22}, a_{33}, \ldots,$ so on until A is diagonalized. Then augmented matrix $\left[A : I\right]$ is transformed to a matrix, say $\left[D : P'\right]$, matrix A is congruent to D.

(vii) The equations of transformation are $X = PY$ and diagonal form of given quadratic form is $d_{11}y_1^2 + d_{22}y_2^2 + \ldots + d_{nn}y_n^2$.

Rank, Index and Signature of a Quadratic Form: The number of non – zero terms in the diagonal form of a quadratic form is called its rank and is denoted by r.

The number of positive terms in the diagonal form of a quadratic form is called its index and is denoted by p.

The number $2p-r$ is called signature of a quadratic form and is denoted by s.

Example 1). Reduce to diagonal form and find the rank, index and signature of the following quadratic forms: $x^2+y^2+4z^2-9t^2-2xy-4yz+6yt-6tx-12tz$.

Solution: The matrix of the given quadratic form is $A=\begin{bmatrix}1 & -1 & 0 & -3\\ -1 & 1 & -2 & 3\\ 0 & -2 & 4 & -6\\ -3 & 3 & -6 & -9\end{bmatrix}$

Consider the augmented matrix $[A:I]=\begin{bmatrix}1 & -1 & 0 & -3 & \vdots & 1 & 0 & 0 & 0\\ -1 & 1 & -2 & 3 & \vdots & 0 & 1 & 0 & 0\\ 0 & -2 & 4 & -6 & \vdots & 0 & 0 & 1 & 0\\ -3 & 3 & -6 & -9 & \vdots & 0 & 0 & 0 & 1\end{bmatrix}$

Operating $R_2 \to R_2 + R_1, R_4 \to R_4 + 3R_1$ $\sim \left[\begin{array}{cccc:cccc} 1 & -1 & 0 & -3 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 4 & -6 & 0 & 0 & 1 & 0 \\ 0 & 0 & -6 & -18 & 3 & 0 & 0 & 1 \end{array}\right]$

Operating $C_2 \to C_2 + C_1, C_4 \to C_4 + 3C_1$, we get $\sim \left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 4 & -6 & 0 & 0 & 1 & 0 \\ 0 & 0 & -6 & -18 & 3 & 0 & 0 & 1 \end{array}\right]$

Operating $R_2 \leftrightarrow R_3, C_2 \leftrightarrow C_3$ we get $\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 4 & -2 & -6 & 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -6 & 0 & -18 & 3 & 0 & 0 & 1 \end{array}\right]$

Operating $R_2 \leftrightarrow \frac{1}{2}R_2, C_2 \leftrightarrow \frac{1}{2}C_2$, we get $\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -3 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -3 & 0 & -18 & 3 & 0 & 0 & 1 \end{array}\right]$

Operating $R_3 \to R_3 + 1R_2, R_4 \to R_4 + 3R_2$ we get $\sim \begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -3 & \vdots & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 & -3 & \vdots & 1 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & -3 & -27 & \vdots & 3 & 0 & \frac{3}{2} & 1 \end{bmatrix}$

Operating $C_3 \to C_3 + 1.C_2,\ C_4 \to C_4 + 3C_2$ we get $\sim \begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \vdots & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 & -3 & \vdots & 1 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & -3 & -27 & \vdots & 3 & 0 & \frac{3}{2} & 1 \end{bmatrix}$

$R_4 \to R_4 + (-3)R_3$, $C_4 \to C_4 + (-3)C_3$, we get

$$\sim \left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 & 0 & 1 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -18 & 0 & -3 & 0 & 1 \end{array}\right] = [D : P']$$

Thus the diagonal form is $y_1^2 + y_2^2 - y_3^2 - 18y_4^2$. and rank = 4, index = 2, signature = 0.

Equations of transformations are given by $X = PY$ i.e. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$

i.e. $x_1 = y_1 + y_3$, $x_2 = y_3 - 3y_4$, $x_3 = \frac{1}{2}y_2 + \frac{1}{2}y_3$ and $x_4 = y_4$.

Canonical Quadratic Form: A quadratic form $q(X) = X'AX$ is said to be canonical if

(i) it is diagonal.

(ii) all non-zero elements on the diagonal of A are either 1 or – 1.

OR

A quadratic form $q(X)$ in the variables $\{x_1, x_2, ..., x_n\}$ is called in canonical form if

$q(X) = \pm x_1^2 \pm x_2^2 \pm ... \pm x_r^2$ where $r \le n$.

Example 2). Reduce the quadratic form $x^2+y^2+4z^2-9t^2-2xy-4yz+6yt-6tx-12tz$ into canonical form and also find the equation of transformation and rank, index, signature.

Solution: For the given quadratic form we already obtained in example (1) that

$$[A:I]=\begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \vdots & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 & 0 & \vdots & 1 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -18 & \vdots & 0 & -3 & 0 & 1 \end{bmatrix}$$

Operating $R_4 \to \frac{1}{\sqrt{18}}R_4, C_4 \to \frac{1}{\sqrt{18}}C_4$, we get $\sim \begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \vdots & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 & 0 & \vdots & 1 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -1 & \vdots & 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{18}} \end{bmatrix}$

Thus the canonical form of quadratic form is $y_1^2 + y_2^2 - y_3^2 - y_4^2$.

Equations of transformations are given by

$$X = PY \text{ i.e. } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{18}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

i.e. $x_1 = y_1 + y_3$, $x_2 = y_3 - \frac{1}{\sqrt{2}} y_4$, $x_3 = \frac{1}{2} y_2 + \frac{1}{2} y_3$ and $x_4 = \frac{1}{\sqrt{18}} y_4$.

Also, rank, index and signature are given by same manner as in the diagonal form.

So Rank = 4, Index = 2 and Signature = 0.

Canonical Quadratic Form Over Complex Field: A quadratic form $q(X) = X'AX$ is said to be in canonical form if matrix A is of the type $A = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$, where I_r denotes the identity matrix of order r.

OR

A quadratic form q(X) in the variables $\{x_1, x_2, \ldots x_n\}$ is called in canonical form over C,

$$q(X) = x_1^2 + x_2^2 + \ldots. + x_r^2, r \le n$$

Example 3). Reduce the quadratic form $x^2+y^2+4z^2-9t^2-2xy-4yz+6yt-6tx-12tz$ into canonical form over complex field and also find the equations of transformation.

Solution: For the given quadratic form we have already obtained in Example 2 that

$$[A:I]\sim\begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \vdots & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 & 0 & \vdots & 1 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -1 & \vdots & 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{18}} \end{bmatrix}$$

Operating $R_3 \to iR_3, C_3 \to iC_3,\ R_4 \to iR_4, C_4 \to iC_4$ we get

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \vdots & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & \vdots & i & i & \frac{1}{2}i & 0 \\ 0 & 0 & 0 & 1 & \vdots & 0 & -\frac{1}{\sqrt{2}}i & 0 & \frac{1}{\sqrt{18}}i \end{bmatrix} = [D : P']$$

Thus the canonical form of quadratic form is $y_1^2 + y_2^2 + y_3^2 + y_4^2$.

Equations of transformations are given by

$$X = PY \text{ i.e. } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & i & 0 \\ 0 & 0 & i & -\frac{1}{\sqrt{2}}i \\ 0 & \frac{1}{2} & \frac{1}{2}i & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{18}}i \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

i.e. $x_1 = y_1 + iy_3$, $x_2 = iy_3 - \frac{1}{\sqrt{2}}iy_4$, $x_3 = \frac{1}{2}y_2 + \frac{1}{2}iy_3$ and $x_4 = \frac{1}{\sqrt{18}}iy_4$.

Lagrange's Method of Diagonalization:

Input: A quadratic form $q(X) = X' AX$.

Output: Diagonal form of $q(X)$.

Working Steps:

(i) If quadratic form is given in the matrix notation then first change it into polynomial expression form. Let the variables used are $x_1, x_2, ..., x_n$.

(ii) Collect the terms containing x_1 in one bracket and take the coefficient of x_1^2 common outside the bracket to make it unity.

(iii) Complete the square by adding and subtracting $\left(\frac{1}{2}\text{coefficient of } x_1\right)^2$.

(iv) Repeat the above process for $x_2, x_3, ...$ and so on.

(v) Give the name $a_1 y_1^2 + a_2 y_2^2 + ... + a_r y_r^2$ to the last obtained expression, this is the required diagonal form and the equations of transformation are given by values of $y_1, y_2, ..., y_r$ in terms of $x_1, x_2, ..., x_n$.

Example 4). Apply Lagrange's method to reduce the following to the diagonal form
$2x^2+2y^2+3z^2-4yz+2xy-4zx$.

Solution: The eigen quadratic form can be expressed as $\left(2x^2+2xy-4xz\right)+2y^2-4yz+3z^2$

$$=2\left[x^2+x(y-2z)\right]+2y^2-4yz+3z^2$$

$$=2\left[x^2+x(y-2z)+\frac{1}{4}(y-2z)^2\right]+2y^2-4yz+3z^2-\frac{1}{2}(y-2z)^2$$

$$=2\left[x+\frac{1}{2}(y-2z)\right]^2+2y^2-4yz+3z^2-\frac{1}{2}(y^2+4z^2-4yz)$$

$$= 2\left[x + \frac{1}{2}(y - 2z)\right]^2 + \frac{3}{2}y^2 - 2yz + z^2$$

$$= 2\left[x + \frac{1}{2}(y - 2z)\right]^2 + \frac{3}{2}\left[y^2 - \frac{4}{3}yz\right] + z^2$$

$$= 2\left[x + \frac{1}{2}(y - 2z)\right]^2 + \frac{3}{2}\left[y^2 - \frac{4}{3}yz + \frac{4}{9}z^2\right] + z^2 - \frac{2}{3}z^2$$

$$= 2\left[x + \frac{1}{2}(y - 2z)\right]^2 + \frac{3}{2}\left[y - \frac{2}{3}z\right]^2 + \frac{1}{3}z^2$$

$$= 2y_1^2 + \frac{3}{2}y_2^2 + \frac{1}{3}y_3^2$$

where $y_1 = x + \frac{1}{2}y - z,\ y_2 = y - \frac{2}{3}z,\ \ y_3 = z$

Assignment - 61

1). Diagonalize the following quadratic forms and also find the rank, index and signature.

(i) $x^2+2y^2-7z^2-4xy+8xz$

(ii) $xy+yz-2xz$

(iii) $X'\begin{bmatrix} 1 & 2 & 3 \\ 2 & -2 & -4 \\ 3 & -4 & -3 \end{bmatrix}X$

(iv) $2x_1^2+2x_2^2+2x_3^2-4x_2x_3-4x_1x_3+2x_1x_2$

(v) $x^2 - 2y^2 + 3z^2 - 4yz + 6zx$

2). Reduce the following quadratic forms into canonical form and also find the equations of transformations. Also find the rank, index and signature.

(i) $x_1^2 + 3x_2^2 + 5x_3^2 + 4x_1x_2 + 4x_1x_3 + 10x_2x_3$

(ii) $X'\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & -4 & 6 & 2 \\ 3 & 6 & 9 & 3 \\ 1 & 2 & 3 & 1 \end{bmatrix}X$

(iii) $2x_1^2 + 2x_2^2 + 3x_3^2 - 4x_2x_3 - 4x_1x_3 + 2x_1x_2$

3). Reduce the following quadratic forms into form $\sum d_i y_i^2$ and also find the equations of transformations.

(i) $2x_1^2 + 2x_2^2 + 3x_3^2 - 4x_2x_3 - 4x_1x_3 + 2x_1x_2$

(ii) $X'\begin{bmatrix} 1 & -2 & 4 \\ -2 & 2 & 0 \\ 4 & 0 & -7 \end{bmatrix}X$

(iii) $x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_3x_4$

4). By using the Lagrange's method diagonalize the following quadratic forms.

(i) $x_1^2+2x_2^2-7x_3^2-4x_1x_2+8x_1x_3$

(ii) $2x_1^2+5x_2^2+19x_3^2-24x_4^2+8x_1x_2+12x_1x_3+8x_1x_4+18x_2x_3-8x_2x_4-16x_3x_4$

5). Reduce the real quadratic form

$x_1^2+3x_2^2+8x_3^2-8x_4^2+4x_1x_2+6x_1x_3+4x_1x_4+10x_2x_3+16x_2x_4+20x_3x_4$

to canonical form over (i) $\mathbb{R}$ (ii) $\mathbb{C}$. Also find the rank and index and write the matrix of transformation.

Answers - Assignment - 61

1). (i) $y_1^2 - 2y_2^2 + 9y_3^2$, rank = 3, index = 2, signature = 1

(ii) $y_1^2 - \frac{1}{4}y_2^2 + 2y_3^2$, rank = 3, index = 2, signature = 1

(iii) $y_1^2 - 6y_2^2 + \frac{14}{3}y_3^2$, rank = 3, index = 2, signature = 1

(iv) $2y_1^2 + \frac{3}{2}y_2^2 + \frac{1}{3}y_3^2$, rank = 3, index = 3, signature = 3

(v) $y_1^2 - 2y_2^2 - 4y_3^2$, rank =3, index = 1, signature = -1

2). (i) $y_1^2 - y_2^2 + y_3^2$, rank = 3, index = 2, signature = 1,

Equations are $x_1 = y_1 - 2y_2 - 2\sqrt{2}y_3$, $x_2 = y_2 + \frac{1}{\sqrt{2}}y_3$, $x_3 = \frac{1}{\sqrt{2}}y_3$

(ii) $y_1^2 - y_2^2$, rank = 2, index = 1, signature = 0,

Equations are $x_1 = y_1 - \frac{y_2}{\sqrt{2}} - 3y_3 - y_4$, $x_2 = \frac{y_2}{\sqrt{2}}$, $x_3 = y_3$, $x_4 = y_4$

(iii) $y_1^2 + y_2^2 + y_3^2$, rank = 3, index = 3, signature = 3, 3). (i) $2y_1^2+\frac{3}{2}y_2^2+\frac{1}{3}y_3^2$, equations are $x_1=y_1-\frac{1}{2}y_2+\frac{2}{3}y_3, x_2=y_2+\frac{2}{3}y_3, x_3=y_3$

(ii) $y_1^2-2y_2^2+9y_3^2$, equations are $x_1=y_1+2y_2+4y_3,\ \ x_2=y_2+4y_3, x_3=y_3$

(iii) $y_1^2-\frac{1}{4}y_2^2-y_3^2-\frac{3}{4}y_4^2$, equations are

$x_1=y_1-\frac{1}{2}y_2-y_3-\frac{1}{2}y_4,\ x_2=y_1+\frac{1}{2}y_2-y_3-\frac{1}{2}y_4,\ x_3=y_3-\frac{1}{2}y_4,\ x_4=y_4$

4). (i) $y_1^2-2y_2^2+9y_3^2$ (ii) $2y_1^2-3y_2^2+4y_3^2$

5). (i) $y_1^2 + y_2^2 - y_3^2$; rank = 3, index = 2; $\begin{bmatrix} 1 & -5 & -2 & -1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 & 0 \end{bmatrix}$

(ii) $y_1^2 + y_2^2 + y_3^2$; $\begin{bmatrix} 1 & -5 & -2i & -1 \\ 0 & 2 & i & -1 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 & 0 \end{bmatrix}$

Definitions of the Quadratic Forms

Positive Definite Quadratic form: A real quadratic form $q(X) = X'AX$ is called positive definite if q is always positive for all non-zero real values of X and is zero only if $X = 0$.

A quadratic form $q(x_1, x_2, ..., x_n)$ is positive definite if in its diagonal form (over reals) all the variables $x_1, x_2, ...x_n$ are present with positive coefficients i.e. $q = a_1x_1^2 + a_2x_2^2 + ... + a_nx_n^2$ where $a_1, a_2, ..., a_n$ are all positive real numbers.

Further in this form we see that rank = index = n i.e. $p = r = n$.

Remark: A real quadratic form $q(X) = X'AX$ is called positive definite if $q(X) > 0, \forall X \neq 0$.

Negative Definite Quadratic Form: A real quadratic form $q = X'AX$ is called negative definite if q is always negative for all non-zero real values of X and is zero only when $X = 0$.

A quadratic form $q(x_1, x_2, ..., x_n)$ is negative definite if in its diagonal form (over reals) all the variables $x_1, x_2, ... x_n$ are present with negative coefficients i.e. $q = -a_1x_1^2 - a_2x_2^2 - ... - a_nx_n^2$ where $a_1, a_2, ..., a_n$ are all positive real numbers. Further in this form we see that rank = n and index = 0 i.e. $r = n$ and $p = 0$.

Remark: A real quadratic form $q = X'AX$ is called negative definite if $q(X) < 0,\ \forall X \neq 0$.

Positive semi-definite Quadratic Form: A real quadratic form $q = X'AX$ is called positive semi-definite if q is always ≥ 0 for all real values of X.

A quadratic form $q(x_1, x_2, ..., x_n)$ is positive semi-definite if in its diagonal form (over reals) at least one variable out of $x_1, x_2, ..., x_n$ is absent and all remaining variables are present with positive coefficients i.e. $q = a_1x_1^2 + a_2x_2^2 + ... + a_nx_n^2,\ r \leq n$, where $a_1, a_2, ..., a_n$ are all positive real numbers. Further in this form we see that

rank = index < n i.e. $p = r < n$.

Remark: A real quadratic form $q = X'AX$ is called positive semi-definite if $q(X) \geq 0, \ \forall X \neq 0$.

Negative Semi-Definite Form: A real quadratic form $q = X'AX$ is called negative semi-definite if q is always ≤ 0 for all real values of X.

A quadratic form $q(x_1, x_2, ..., x_n)$ is negative semi-definite if in its diagonal form (over reals) at least one variable out of $x_1, x_2, ..., x_n$ is absent and all remaining variables are present with negative coefficients i.e. $q = -a_1x_1^2 - a_2x_2^2 - ... - a_nx_n^2, \ r \leq n$,

where $a_1, a_2, ..., a_n$ are all positive real numbers.

Further in this form we see that rank $< n$ and index $= 0$ i.e. $r < n$ and $p = 0$.

Remark: A real quadratic form $q = X'AX$ is called negative semi-definite if $q(X) \le 0, \forall X \ne 0$.

Indefinite Quadratic Form: A real quadratic form $q = X'AX$ is called indefinite if q can assume both positive and negative values for different real values of X.

A quadratic form $q(x_1, x_2, ..., x_n)$ is indefinite if in its diagonal form (over reals) at least one negative and one positive term is present.

Remark: A real quadratic form $q = X'AX$ is called indefinite if $q(X_1) > 0$ for some $X_1 \ne 0$, $q(X_2) < 0$ for some $X_2 \ne 0$.

Def. Let $q = X'AX$ be a real quadratic form. The definiteness of the matrix A is defined to be the definiteness of the quadratic form q.

Range of a Quadratic Form: The set of all values attained by a quadratic form is called its range.

Results: (i) Range of a positive definite and positive semi definite quadratic form is $[0, \infty)$.

(ii) Range of a negative definite and negative semi definite quadratic form is $(-\infty, 0]$.

(iii) Range of an indefinite quadratic form is $(-\infty, \infty)$.

Example: Determine the definiteness of the quadratic form

$6x^2 + 25y^2 + 61z^2 - 24xy - 36xz + 76yz$.

Solution: The matrix of the given quadratic form is $A = \begin{bmatrix} 6 & -12 & -18 \\ -12 & 25 & 38 \\ -18 & 38 & 61 \end{bmatrix}$

We have already obtained the diagonal form of this quadratic form in Example 4, namely, $6y_1^2 + y_2^2 + 3y_3^2$. Clearly this canonical form contains all the three variables and also with positive coefficients, therefore the given quadratic form positive definite.

Note: If a quadratic form contains atleast one square term with positive sing and one square term with negative sign, then that quadratic form is always indefinite.

Assignment - 62

1). Determine the definiteness of the following quadratic forms using the diagonalization process.

(i) $6x^2+3y^2+3z^2-4xy-2yz+4zx$

(ii) $9x^2+y^2+4z^2+6xy-4yz-12zx$

(iii) $-21x^2-11y^2-2z^2+30xy+8yz-12zx$

(iv) $-y^2-2xy+2yz$

(v) $2x_1^2+4x_2^2+8x_3^2-2x_2x_3-2x_1x_3+2x_1x_2$

(vi) $x^2 - 2y^2 + 3z^2 - 4yz + 6zx$

(vii) $-3x^2 - 4y^2 + 2z^2 - 2yz + xz + 7xy$

Answers - Assignment - 62

1). (i) Positive Definite

(ii) Positive Semi definite

(iii) Negative Semi definite

(iv) Indefinite

(v) Positive definite

(vi) Indefinite

(vii) Indefinite

Sylvester's Criterion

Example: Write down all the minors, principal minors and the leading principal minor of order 1, 2, 3 for a 3×3 matrix.

Solution: Let $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

(i) Minors of order 1: $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$.

Principal minors of order 1: a_{11}, a_{22}, a_{33}

Leading Principal minor of order 1: a_{11}

(ii) Minors order of 2: $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{vmatrix},$

$$\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}, \begin{vmatrix} a_{12} & a_{13} \\ a_{23} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Principal minors or order 2: $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{21} & a_{22} \\ a_{32} & a_{23} \end{vmatrix}$

Leading Principal minor of order 2: $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

(iii) Minor of order 3: $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

There is only one minor of order 3 which acts as principal minor and leading principal minor as well.

Result: Let $A = \left[a_{ij}\right]_{n\times n}$ be a matrix, then

(i) Number of all minors of order $r\times r$ is $\left({}^{n}C_{r}\right)^{2}$

(ii) Number of all principal minors of order $r\times r$ is ${}^{n}C_{r}$.

(iii) Number of leading principal minor of $r \times r$ is 1.

Theorem 1). A quadratic form $q(X) = X^T AX$ is positive definite iff all the leading principal minors of A are positive.

Particular Cases:

(i) The matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is positive definite iff $a_{11} > 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$

(ii) The matrix $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is positive definite iff

$$a_{11}>0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}>0, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}>0$$

Theorem 2). A quadratic form $q(X)=X^{T}AX$ is negative definite iff all the leading principal minors are alternatively negative and positive, starting with first.

Particular Cases:

(i) The matrix $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is negative definite iff $a_{11}<0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}>0$

(ii) The matrix $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is negative definite iff

$$a_{11}<0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}>0, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}<0$$

Theorem 3). A quadratic form $q(X) = X^T AX$ is positive semi definite iff A is singular i.e. $|A| = 0$ and all other principal minors are non-negative.

Particular Cases:

(i) The matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is positive semi-definite iff $a_{11} \geq 0$, $a_{22} \geq 0$ and

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0 .$$

(ii) The matrix $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is positive semi-definite iff

$a_{11}\geq 0,\ a_{22}\geq 0,\ a_{33}\geq 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\geq 0, \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}\geq 0, \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}\geq 0$ and $|A|=0$.

Theorem 4). A quadratic form $q(X)=X^{T}AX$ is negative semi definite iff $|A|=0$ and all principal minors of ever order of A are non-negative while those of odd order are non-positive.

Particular Cases:

(i) The matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is negative semi-definite iff $a_{11} \le 0$ and $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0$

(ii) The matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is negative semi-definite if $a_{11} \le 0, a_{22} \le 0, a_{33} \le 0$,

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \ge 0$, $\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \ge 0$, $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \ge 0$ and $|A| = 0$.

Theorem 5). A quadratic form $q(X) = X^T AX$ is indefinite iff at least one of the following condition is satisfied.

(i) A has negative principal minor of even order.

(ii) A has positive principal minor of odd order and negative minor of ever order.

Particular Cases:

(i) The matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is indefinite iff $|A| < 0$ and a_{11} and a_{22} are of opposite signs.

(ii) The matrix $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is indefinite iff

(a) $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} < 0$ or $\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} < 0$ or $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} < 0$

(b) Out of $a_{11}, a_{22}, a_{33}, |A|$ at least one is positive and at least one is negative.

Theorem 6). A quadratic form $q(X) = X^T AX$ is positive semi-definite if $|A| = 0$ and the other leading principal minors are positive.

Particular Cases:

(i) The matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is positive semi-definite if $a_{11} > 0$, $\det(A) = 0$

(ii) The matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is positive semi-definite if $a_{11} > 0$, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$ and $\det(A) = 0$.

Theorem 7). A quadratic form $q(X) = X^T AX$ is negative semi-definite if $\det(A) = 0$ and the other Principal Minor of odd order are negative and of even order are positive.

Eigen Value Theorem for Definitions: A quadratic form $q(X) = X^T AX$ is

(i) Positive definite iff all eigen value of A are positive.

(ii) Negative definite iff all eigen value of A are negative.

(iii) Positive semi-definite iff at-least one eigen value is zero and all other eigen values are non-negative.

(iv) Negative semi-definite iff at-least one eigen value is zero and all other eigen values are non-positive.

(v) Indefinite iff at-least one eigen value is positive and at-least one eigen value is negative.

Results:

1). If A and B are two congruent matrices then they have same rank, same index and same signature.

OR

If two matrices are congruent then they have same number of positive eigen values and same number of negative eigen values.

2). If A and B are two positive definite matrices, then A + B is also positive definite.

Proof: If A and B are positive definite, then for all non-zero values of X, we have

$$X'AX > 0 \text{ and } X'BX > 0$$

$$\Rightarrow X'AX + X'BX > 0$$

$$\Rightarrow X'(A+B)X > 0$$

$\Rightarrow$ $A+B$ is positive definite.

3). If A and B are two positive definite matrices such that $AB = BA$, then AB is also positive definite.

4). If A is positive definite matrix, then A^k is also positive definite for all $k \in \mathbb{Z}$.

Proof: If A is positive definite, then all the eigen values of A are positive and hence all the eigen values of A^k are also positive. Therefore, A^k is positive definite.

5). If A and B are two positive definite matrices, then ABA is also positive definite.

Proof: Since A is positive definite, therefore A is symmetric also i.e. $A' = A$

$$\Rightarrow \quad ABA = A'BA \equiv B$$

Thus, $A'BA$ and B have same number of positive eigen values and B is positive definite, therefore $A'BA = ABA$ is also positive definite.

6). If A is positive definite matrix, then $A'A$ is also positive definite.

Proof: Since A is positive definite, therefore A is symmetric also i.e. $A' = A$

$$\Rightarrow A'A = A^2$$

And A^2 is positive definite by result (4). Therefore, $A'A$ is positive definite.

Assignment - 63

1). Determine the definiteness of the following quadratic forms using the Sylvester's Criterion.

(i) $6x^2 + 3y^2 + 14z^2 + 4xy + 4yz + 18zx$

(ii) $x^2 + y^2 + z^2 - xy + yz - zx$

(iii) $-x^2 - 2y^2 - 2z^2 + 2xy + 2yz$

(iv) $-x^2 - 3y^2 - 2z^2$

(v) $-21x_1^2 + 30x_1x_2 - 12x_1x_3 - 11x_2^2 + 8x_2x_3 + 2x_3^2$

2). Prove that the quadratic form $ax^2 + 2hxy + by^2$ is positive definite iff $a > 0$ & $h^2 < ab$.

3). For a positive definite quadratic for $X'AX$, prove that $|A| > 0$.

Answers - Assignment - 63

1). (i) Positive Definite

(ii) Positive definite

(iii) Negative definite

(iv) Negative definite

(v) Indefinite

Assignment (SCQ) - 64

1). Consider the quadratic form $Q(v) = v^t Av$, where $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $v = (x, y, z, w)$

Then

1. Q has rank 3
2. $xy + z^2 = Q(Pv)$ for some invertible 4×4 real matrix P.
3. $xy + y^2 + z^2 = Q(Pv)$ for some invertible 4×4 real matrix P.
4. $x^2 + y^2 - zw = Q(Pv)$ for some invertible 4×4 real matrix P.

(CSIR NET Dec 2015)

2). Let A be a $n \times n$ real matrix symmetric non-singular matrix. Suppose there exists $x \in \mathbb{R}^n$ such that $x' Ax < 0$. Then we can conclude that

1. $\det(A) < 0$
2. $B = -A$ is positive definite.
3. $\exists y \in \mathbb{R}^n;\ y' A^{-1} y < 0$
4. $\forall y \in \mathbb{R}^n;\ y' A^{-1} y < 0$

(CSIR NET June 2016)

3). Let $A=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$. Let $f:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$ be defined by $f(v,w)=w^T Av$.

Pick the correct statement from below:

1. There exists an eigenvector v of A such that Av is perpendicular to v.
2. The set $\{v\in\mathbb{R}^2 \mid f(v,v)=0\}$ is a non-zero subspace of $\mathbb{R}^2$
3. If $v,w\in\mathbb{R}^2$ are non-zero vectors such that $f(v,v)=0=f(w,w)$, then v is a scalar multiple of w.
4. For every $v\in\mathbb{R}^2$, there exists a non-zero $w\in\mathbb{R}^2$ such that $f(v,w)=0$.

(CSIR NET June 2016)

4). The matrix $\begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$ is

1. positive definite.
2. non-negative definite but not positive definite.
3. negative definite.
4. neither negative definite nor positive definite.

(CSIR NET Dec 2016)

5). Let a, b, c be positive real numbers such that $b^2 + c^2 < a < 1$. Consider the 3×3 matrix

$$A = \begin{bmatrix} 1 & b & c \\ b & a & 0 \\ c & 0 & 1 \end{bmatrix}.$$

1. All the eigenvalues of A are negative real numbers
2. All eigenvalues of A are positive real numbers
3. A can have a positive as well as a negative eigenvalue
4. Eigenvalue of A can be non real complex numbers.

(CSIR NET Dec 2012)

6). Let $A=\begin{bmatrix}1 & 2\\4 & 3\end{bmatrix}\in M_2(\mathbb{R})$ and $\phi:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$ be the bilinear map defined by $\phi(v,w)=v^T Aw$. Choose the correct statement from below:

1. $\phi(v,w)=\phi(w,v)$ for all $v,w\in\mathbb{R}^2$
2. there exists nonzero $v\in\mathbb{R}^2$ such that $\phi(v,w)=0$ for all $w\in\mathbb{R}^2$
3. there exists a 2 × 2 symmetric matrix B such that $\phi(v,v)=v^T Bv$ for all $v\in\mathbb{R}^2$
4. the map $\psi:\mathbb{R}^4\to\mathbb{R}$ defined by $\psi\left(\begin{bmatrix}v_1\\v_2\\w_1\\w_2\end{bmatrix}\right)=\phi\left(\begin{bmatrix}v_1\\v_2\end{bmatrix},\begin{bmatrix}w_1\\w_2\end{bmatrix}\right)$ is linear

(CSIR NET Dec 2017)

7). Let $B:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be the function $B(a,b)=a,b$. Which of the following is true?

1. B is linear transformation.
2. B is a positive definite bilinear form
3. B is symmetric but not positive definite
4. B is neither linear nor bilinear

(CSIR NET Dec 2018)

Assignment (MCQ) - 65

1). Let $a_{ij} = a_i a_j, 1 \le i, j \le b$ where $a_1, \ldots a_n$ are real numbers. Let $A = ((a_{ij}))$ be the $n \times n$ matrix $((a_{ij}))$. Then

1. It is possible to choose $a_1, \ldots, a_n$ so as to make the matrix A non-singular.
2. The matrix A is positive definite if $(a_1, \ldots, a_n)$ is a non-zero vector.
3. The matrix A is positive semidefinite for all $(a_1, \ldots, a_n)$.
4. For all $(a_1, \ldots, a_n)$, zero is an eigen value of A.

(CSIR NET June 2011)

2). Suppose A, B are $n \times n$ positive definite matrices and I be the $n \times n$ identity matrix. Then which of the following are positive definite

1. $A + B$
2. ABA
3. $A^2 + I$
4. AB

(CSIR NET June 2011)

3). Which of the following matrices are positive definite?

1. $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$

(CSIR NET June 2012)

4). Suppose A is a 3×3 symmetric matrix such that $[x,y,1]A\begin{bmatrix}x\\y\\1\end{bmatrix}=xy-1$.

Let p be the number of positive eigenvalues of A and let $q=\text{rank}(A)-p$. Then

1. $p=1$
2. $p=2$
3. $q=2$
4. $q=1$

(CSIR NET Dec 2012)

5). Let A be a $n \times n$ non-singular matrix with real matrices. Let $B = A^T$ denote the transpose of A. Which of the following matrices are positive definite?

1. $A + B$
2. $A^{-1} + B^{-1}$
3. AB
4. ABA

(CSIR NET Dec 2016)

6). Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ and $Q(X) = X^t AX$ for $X \in \mathbb{R}^3$. Then

1. A has exactly two positive eigenvalues
2. all the eigenvalue of A are positive
3. $Q(X) \geq 0$ for all $x \in \mathbb{R}^3$
4. $Q(X) < 0$ for some $x \in \mathbb{R}^3$

(CSIR NET June 2017)

7). Consider the quadratic form $q(x,y,z)=4x^2+y^2-z^2+4xy-2xz-yz$ over $\mathbb{R}$. Which of the following statements about the range of values taken by q as x, y, z varies over $\mathbb{R}$, are true?

1. range contains $[1,\infty)$
2. range is contained in $[0,\infty)$
3. range $=\mathbb{R}$
4. range is contained in $[-N,\infty)$ for some large natural number N depending on q.

(CSIR NET Dec 2011)

8). Which of the following matrices are positive definite?

1. $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$

(CSIR NET June 2012)

9). Consider the quadratic form q and p given by $q(x,y,z,w)=x^2+y^2+z^2+bw^2$ and $p(x,y,z,w)=x^2+y^2+czw$.

Which of the following statements are true?

1. p and q are equivalent over $\mathbb{C}$ if b and c are non-zero complex numbers.
2. p and q are equivalent over $\mathbb{R}$ if b and c are non-zero real numbers.
3. p and q are equivalent over $\mathbb{R}$ if b and c are non-zero real numbers with b negative.
4. p and q are NOT equivalent over $\mathbb{R}$ if $c = 0$

(CSIR NET June 2013)

10). For any real square matrix M let $\lambda^{+}(M)$ be the number of positive eigenvalue of M counting multiplicities. Let A be an $n \times n$ real symmetric matrix and Q be an $n \times n$ real invertible matrix. Then

1. $\text{Rank}A = \text{Rank}Q^{T}AQ$
2. $\text{Rank}A = \text{Rank}Q^{-1}AQ$
3. $\lambda^{+}(A) = \lambda^{+}(Q^{T}AQ)$
4. $\lambda^{+}(A) = \lambda^{+}(Q^{-1}AQ)$

(CSIR NET Dec 2013)

11). Let $f:\mathbb{R}^4 \to \mathbb{R}$ be defined by $f(x) = x^t Ax$, where A is a 4×4 matrix with real entries and x^t denotes the transpose of x. The gradient of f at a point x_0 necessarily is

1. $2Ax_0$
2. $Ax_0 + A'x_0$
3. $2A^t x_0$
4. Ax_0

(CSIR NET Dec 2017)

12). For every 4×4 real symmetric non-singular matrix A, there exists a positive integer p such that

1. $pI + A$ is positive definite
2. A^p is positive definite
3. A^{-p} is positive definite
4. $\exp(pA) - I$ is positive definite

(CSIR NET Dec 2017)

13). Let $A=\begin{pmatrix}1 & 2 & 0\\ 0 & 0 & -2\\ 0 & 0 & 1\end{pmatrix}$ and define for $x, y, z \in \mathbb{R}$ $Q(x,y,z)=(x,y,z)A\begin{pmatrix}x\\ y\\ z\end{pmatrix}$.

Which of the following statements are true?

1. The matrix of second order partial derivatives of the quadratic form Q is $2A$
2. The rank of the quadratic form Q is 2
3. The signature of the quadratic form Q is $(++0)$
4. The quadratic form Q takes the value 0 for some non-zero vector (x,y,z)

(CSIR NET June 2018)

14). Define a real valued function B on $\mathbb{R}^2 \times \mathbb{R}^2$ as follows, If $v = (x_1, x_2)$, $w = (y_1, y_2)$ belong to $\mathbb{R}^2$ define $B(u, w) = x_1 y_1 - x_1 y_2 - x_2 y_1 + 4x_2 y_2$. Let $v_0 = (1,0)$ and let $W = \{v \in \mathbb{R}^2 : B(v_0, v) = 0\}$. Then W

1. is not a subspace of $\mathbb{R}^2$

2. equals $\{(0,0)\}$

3. is the y –axis

4. is the line passing trough (0, 0) and (1, 1)

(CSIR NET Dec 2018)

15). Consider the Quadratic forms

$$Q_1(x,y) = xy$$
$$Q_2(x,y) = x^2 + 2xy + y^2$$
$$Q_3(x,y) = x^2 + 3xy + 2y^2$$

on $\mathbb{R}^2$. Choose the correct statements from below;

1. Q_1 and Q_2 are equivalent
2. Q_1 and Q_3 are equivalent
3. Q_2 and Q_3 are equivalent
4. All are equivalent

(CSIR NET Dec 2018)

16). Consider a matrix $A=\left(a_{ij}\right)_{5\times5}$, $1\le i,j\le 5$ such that $a_{ij}=\dfrac{1}{n_i+n_j+1}$, where $n_i,n_j\in\mathbb{N}$.

Then in which of the following cases A is a positive definite matrix?

1. $n_i=i$ for all $i=1,2,3,4,5$

2. $n_1<n_2<\ldots<n_5$

3. $n_1=n_2=\ldots=n_5$

4. $n_1>n_2>\ldots>n_5$

(CSIR NET June 2019)

Assignment Key (SCQ) - 64

1. 4	2. 3	3. 4	4. 1
5. 2	6. ---		

Assignment Key (SMQ) - 65

1. 3, 4	2. 1, 2, 3	3. 1, 3	4. 1, 3
5. 3	6. 1, 4	7. 1, 3	8. 1, 3
9. 1, 3, 4	10. 1,2,3,4	11. ---	12. 1, 2, 3
13. 4			

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